

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Week 11 Worksheet - last worksheet on Taylor Polynomial

Instructions. Follow the instructions of your TA and do the following problems. You are not expected to finish all the problems. So take your time! :)

Today's topic: Taylor Polynomial - Multiplication, differentiation, integration.

1. Compute $T_3(\sin x \cos x)$

2 ways.

$$\textcircled{1} \quad \sin x \cos x$$

$$= \left[x - \frac{x^3}{3!} + o(x^3) \right] \cdot \left[1 - \frac{x^2}{2!} + o(x^3) \right]$$

$$= x - \frac{x^3}{2} - \frac{x^3}{6} + o(x^3)$$

$$\textcircled{2} \quad \sin x \cos x = \frac{1}{2} \sin(2x)$$

$$= \frac{1}{2} \left[2x - \frac{(2x)^3}{3!} + o(x^3) \right]$$

$$= x - \frac{2}{3} x^3 + o(x^3)$$

$$\Rightarrow T_3(\sin x \cos x) = x - \frac{2}{3} x^3$$

$$T_3[\sin x \cos x] = x - \frac{2}{3} x^3$$

2. Compute $T_4[e^{-t} \sin(2t)]$.

$$e^{-t} \sin(2t)$$

$$= \left[1 - t + \frac{(-t)^2}{2!} + \frac{(-t)^3}{3!} + \frac{(-t)^4}{4!} + o(t^4) \right] \left[2t - \frac{(2t)^3}{3!} + o(t^4) \right]$$

$$= \left[1 - t + \frac{1}{2} t^2 - \frac{1}{6} t^3 + o(t^3) \right] \left[2t - \frac{4}{3} t^3 + o(t^4) \right]$$

$$= 2t - 2t^2 + t^3 - \frac{1}{3} t^4 - \frac{4}{3} t^3 + \frac{4}{3} t^4 + o(t^4)$$

$$\Rightarrow T_4[e^{-t} \sin(2t)] = 2t - 2t^2 - \frac{1}{3} t^3 + t^4$$

3. Compute the Taylor Series of $\arctan(2x)$.

$$[\arctan(ax)]' = \frac{2}{1+4x^2} = 2 \cdot \frac{1}{1+4x^2} \quad \text{make } u = -4x^2 \text{ use formula of } \frac{1}{1-u}$$

$$= 2 \cdot \sum_{n=0}^{\infty} u^n = 2 \cdot \sum_{n=0}^{\infty} (-4x^2)^n = 2 \sum_{n=0}^{\infty} (-4)^n x^{2n}$$

$$\Rightarrow \arctan(ax) = \int 2 \sum_{n=0}^{\infty} (-4)^n x^{2n} dx$$

$$= 2 \sum_{n=0}^{\infty} (-4)^n \int x^{2n} dx$$

$$= 2 \sum_{n=0}^{\infty} (-4)^n \frac{x^{2n+1}}{2n+1} + C$$

$$\text{Plug in } x=0 \Rightarrow C=0$$

$$\Rightarrow \arctan(ax) = \sum_{n=0}^{\infty} \frac{2 \cdot (-4)^n}{2n+1} x^{2n+1}$$

4. (from 2013final) Find the Taylor Polynomial of degree 14 at $x = 0$ (i.e. find T_{14}) of the function $f(x) = \frac{10x^4}{(1-x^5)^2}$.

$$\int f(x) dx = \int \frac{10x^4}{(1-x^5)^2} dx \quad \begin{matrix} u = 1-x^5 \\ du = -5x^4 dx \end{matrix} \quad \int \frac{-2 du}{u^2} = \int -2u^{-2} du = -2 \frac{u^{-1}}{-1} + C = \frac{2}{u} + C$$

$$\frac{2}{1-x^5} = 2 \cdot \frac{1}{1-x^5} = 2 \sum_{n=0}^{\infty} (x^5)^n = 2 \sum_{n=0}^{\infty} x^{5n}$$

$$f(x) = \left(2 \sum_{n=0}^{\infty} x^{5n} \right)' = 2 \sum_{n=1}^{\infty} 5n \cdot x^{5n-1}$$

$$\Rightarrow T_{14}(x) = 10x^4 + 100x^9 + 70x^{14}$$

5. (a) Find the second degree Taylor polynomial for the function e^t .

(b) Use it to give an estimate for the integral

$$\int_0^1 e^{x^2} dx.$$

the error using

(c) Give an upper bound of this approximation for $\int_0^1 e^{x^2} dx$.

$$(a) T_2 e^t = 1 + t + \frac{t^2}{2}$$

$$(b) e^{x^2} \approx 1 + x^2 + \frac{x^4}{2} \quad (\text{use } t = x^2)$$

$$\int_0^1 e^{x^2} dx \approx \int_0^1 1 + x^2 + \frac{x^4}{2} dx = x + \frac{x^3}{3} + \frac{x^5}{10} \Big|_{x=0}^{x=1} = 1 + \frac{1}{3} + \frac{1}{10}$$

$$(c) \text{Error} = \int_0^1 \left| e^{x^2} - (1 + x^2 + \frac{x^4}{2}) \right| dx \quad \text{Notice } |e^t - T_2 e^t| = |T_2 e^t| = \left| \frac{f^{(3)}(\zeta)}{3!} t^3 \right| \quad \begin{matrix} \zeta \text{ is between} \\ 0 \text{ and } t \end{matrix}$$

$$= \int_0^1 \frac{e^{\zeta}}{6} x^6 dx \quad (e^{\zeta} \leq e^1 \leq 3) \quad = \left| \frac{e^{\zeta}}{6} t^3 \right| \xrightarrow{t=x^2} \frac{e^{\zeta}}{6} x^6$$

$$\leq \int_0^1 \frac{3}{6} x^6 dx = \frac{1}{2} \frac{x^7}{7} \Big|_{x=0}^{x=1} = \frac{1}{14}$$

6. Compute the Taylor Series of the following functions:

$$(a) \frac{1}{\sqrt{1-t}}$$

$$(a) \frac{1}{\sqrt{1-t}} = (1-t)^{-\frac{1}{2}}$$

use formula of $(1+x)^a$

$$(b) \frac{1}{\sqrt{1-t^2}}$$

$$(c) \arcsin t$$

$$= \sum_{n=0}^{\infty} \binom{a}{n} (-t)^n = \sum_{n=0}^{\infty} (-1)^n \binom{a}{n} t^n$$

(b) Use the result of (a), replace t by t^2 .

$$\frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} (-1)^n \binom{a}{n} t^{2n}$$

$$(c) (\arcsin t)' = \frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} (-1)^n \binom{a}{n} t^{2n-1}$$

$$\Rightarrow \arcsin t = \sum_{n=0}^{\infty} (-1)^n \binom{a}{n} \int t^{2n-1} dt = \sum_{n=0}^{\infty} (-1)^n \binom{a}{n} \frac{t^{2n+1}}{2n+1} + C$$

$$\text{Plug in } t=0 \Rightarrow C=0 \quad \text{So in sum, } \arcsin t = \sum_{n=0}^{\infty} (-1)^n \binom{a}{n} \frac{t^{2n+1}}{2n+1}$$